Louisiana Department of Education
Mentor Teacher Training

Module 7:
Delivering High-Quality, Standards-Aligned
Math Instruction

Secondary Math Cohort

Facilitated by Learning Forward

Mentor Training Goals

1. Build strong relationships with their new and resident teachers.
2. Use quantitative data (i.e. student achievement results from summative and formative assessments) and qualitative data (i.e. student work and feedback, classroom observations, peer and supervisor feedback) to identify and prioritize new and resident teacher's strengths and areas for growth based on teacher **certification competencies and performance expectations**.
3. Implement a continuum of support and coaching techniques to build new and resident teacher knowledge and skill in content and content-specific pedagogy and self-analysis and reflection skills.
   - Set short- and long-term professional growth goals based on student results.
   - Create a logical sequence of coaching opportunities to build new/resident teacher's skill over time to reach goals.
   - Identify and leverage high-quality, evidence-based resources to support new and resident teachers' specific needs.
   - Lead effective feedback/coaching conversations grounded in student evidence (includes difficult conversations).
   - Track new and resident teacher progress.
   - Model and facilitate resident self-reflection and improvement.
4. Deepen content knowledge and content-specific pedagogy to support continued development of teacher competencies in the respective content areas.
5. Assess and develop mentor competencies in their discipline to identify strengths and gaps and to develop areas of need. These assessments are based on competencies and student results.

Module 7 Outcomes

- Provide evidence of focus, coherence, and rigor in a model EngageNY lesson.
- Closely examine modeling with mathematics and recognize its critical role in the development of students' mathematical thinking.
- Promote students' persistence and effective effort in the classroom.
- Gain clarity on how the mathematics standards manifest in the secondary classroom when executed well in curriculum instruction.

Module 7 Key Points

- Mentors, to succeed in supporting resident development, have deep content knowledge and content-specific pedagogy.
- The key shifts in the standards of focus, coherence, and rigor are evident in an EngageNY lesson and are supported by the content and practice standards.
- Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, understand the situations better, and to improve precision and processes for decision---making.
- Mentors recognize and describe strategic opportunities for students to apply persistence and effective effort strategies in the EngageNY lessons.

Module 7 Agenda

8:30-12:00

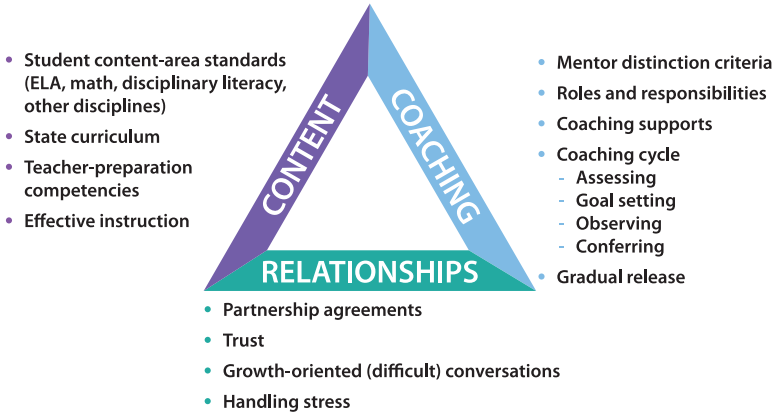
- Review and revisit Modules 5 & 6
- Reflection on assignments
- Key shifts in action through an EngageNY lesson
- Instructional strategies to improve curriculum implementation

12:45-4:00

- The role of a productive classroom culture
- Key shifts in action through an EngageNY lesson
- Teacher Preparation Competencies in mathematics

Agreements

- **M**ake the learning meaningful
- **E**ngage mentally and physically
- **N**otice opportunities to support the learning of others
- **T**ake responsibility for your own learning
- **O**wn the outcomes
- **R**espect the learning environment and the use of personal technology



Reflections on Module 5-6 assignments

Partner task:

Share your case study and discuss your analysis of the teacher. (5 minutes per partner)

Table task:

Share your analysis of a professional learning experience. Note patterns across your team of standards present or absent in your individual experience. (10 minutes)

Table of Contents[1]

Ratios and Proportional Relationships

Module Overview 2

Topic A: Proportional Relationships (**7.RP.A.2a**) 10

Lesson 1: An Experience in Relationships as Measuring Rate 11

Lesson 2: Proportional Relationships 19

Lessons 3–4: Identifying Proportional and Non-Proportional Relationships in Tables 26

Lessons 5–6: Identifying Proportional and Non-Proportional Relationships in Graphs 41

Topic B: Unit Rate and the Constant of Proportionality (**7.RP.A.2b**, **7.RP.A.2c**, **7.RP.A.2d**, **7.EE.B.4a**) 58

Lesson 7: Unit Rate as the Constant of Proportionality 60

Lessons 8–9: Representing Proportional Relationships with Equations 67

Lesson 10: Interpreting Graphs of Proportional Relationships 86

Mid-Module Assessment and Rubric 95
Topics A through B (assessment 1 day, return 1 day, remediation or further applications 2 days)

Topic C: Ratios and Rates Involving Fractions (**7.RP.A.1**, **7.RP.A.3**, **7.EE.B.4a**) 103

Lessons 11–12: Ratios of Fractions and Their Unit Rates 105

Lesson 13: Finding Equivalent Ratios Given the Total Quantity 119

Lesson 14: Multi-Step Ratio Problems 128

Lesson 15: Equations of Graphs of Proportional Relationships Involving Fractions 135

Topic D: Ratios of Scale Drawings (**7.RP.A.2b**, **7.G.A.1**) 143

Lesson 16: Relating Scale Drawings to Ratios and Rates 144

Lesson 17: The Unit Rate as the Scale Factor 157

Lesson 18: Computing Actual Lengths from a Scale Drawing 167

Lesson 19: Computing Actual Areas from a Scale Drawing 177

Lesson 20: An Exercise in Creating a Scale Drawing 187

Lessons 21–22: An Exercise in Changing Scales 196

End-of-Module Assessment and Rubric 211
Topics A through D (assessment 1 day, return 1 day, remediation or further applications 2 days)

[1]Each lesson is ONE day, and ONE day is considered a 45-minute period.

© 2015 Great Minds. eureka-math.org
G7-M1-TE-1.3.0-06.2015

This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Grade 7 • Module 1

Ratios and Proportional Relationships

OVERVIEW

In Module 1, students build upon their Grade 6 reasoning about ratios, rates, and unit rates (**6.RP.A.1**, **6.RP.A.2**, **6.RP.A.3**) to formally define proportional relationships and the constant of proportionality (**7.RP.A.2**). In Topic A, students examine situations carefully to determine if they are describing a proportional relationship. Their analysis is applied to relationships given in tables, graphs, and verbal descriptions (**7.RP.A.2a**).

In Topic B, students learn that the unit rate of a collection of equivalent ratios is called the *constant of proportionality* and can be used to represent proportional relationships with equations of the form $y = kx$, where k is the constant of proportionality (**7.RP.A.2b**, **7.RP.A.2c**, **7.EE.B.4a**). Students relate the equation of a proportional relationship to ratio tables and to graphs and interpret the points on the graph within the context of the situation (**7.RP.A.2d**).

In Topic C, students extend their reasoning about ratios and proportional relationships to compute unit rates for ratios and rates specified by rational numbers, such as a speed of $\frac{1}{2}$ mile per $\frac{1}{4}$ hour (**7.RP.A.1**). Students apply their experience in the first two topics and their new understanding of unit rates for ratios and rates involving fractions to solve multi-step ratio word problems (**7.RP.A.3**, **7.EE.B.4a**).

In the final topic of this module, students bring the sum of their experience with proportional relationships to the context of scale drawings (**7.RP.A.2b**, **7.G.A.1**). Given a scale drawing, students rely on their background in working with side lengths and areas of polygons (**6.G.A.1**, **6.G.A.3**) as they identify the scale factor as the constant of proportionality, calculate the actual lengths and areas of objects in the drawing, and create their own scale drawings of a two-dimensional view of a room or building. The topic culminates with a two-day experience of students creating a new scale drawing by changing the scale of an existing drawing.

Later in the year, in Module 4, students extend the concepts of this module to percent problems.

The module is composed of 22 lessons; 8 days are reserved for administering the Mid- and End-of-Module Assessments, returning the assessments, and remediating or providing further applications of the concepts. The Mid-Module Assessment follows Topic B. The End-of-Module Assessment follows Topic D.

© 2015 Great Minds. eureka-math.org
G7-M1-TE-1.3.0-06.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Focus Standards

Analyze proportional relationships and use them to solve real-world and mathematical problems.

7.RP.A.1 Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. *For example, if a person walks* $1/2$ *mile in each* $1/4$ *hour, compute the unit rate as the complex fraction* ${}^{1/2}/_{1/4}$ *miles per hour, equivalently* 2 *miles per hour.*

7.RP.A.2 Recognize and represent proportional relationships between quantities.

a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.

b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.

c. Represent proportional relationships by equations. *For example, if total cost* t *is proportional to the number* n *of items purchased at a constant price* p*, the relationship between the total cost and the number of items can be expressed as* $t = pn$.

d. Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points $(0,0)$ and $(1, r)$, where r is the unit rate.

7.RP.A.3 Use proportional relationships to solve multistep ratio and percent problems. *Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.*

Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

7.EE.B.4[2] Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

a. Solve word problems leading to equations of the form $px + q = r$ and $p(x + q = r$, where p, q, and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. *For example, the perimeter of a rectangle is* 54 *cm. Its length is* 6 *cm. What is its width?*

[2]In this module, the equations are derived from ratio problems. 7.EE.B.4a is returned to in Modules 2 and 3.

© 2015 Great Minds. eureka-math.org
G7-M1-TE-1.3.0-06.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Draw, construct, and describe geometrical figures and describe the relationships between them.

7.G.A.1 Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.

Foundational Standards

Understand ratio concepts and use ratio reasoning to solve problems.

6.RP.A.1 Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities. *For example, "The ratio of wings to beaks in the bird house at the zoo was* $2:1$*, because for every* 2 *wings there was* 1 *beak." "For every vote candidate A received, candidate C received nearly three votes."*

6.RP.A.2 Understand the concept of a unit rate a/b associated with a ratio $a:b$ with $b \neq 0$, and use rate language in the context of a ratio relationship. *For example, "This recipe has a ratio of* 3 *cups of flour to* 4 *cups of sugar, so there is* $3/4$ *cup of flour for each cup of sugar." "We paid* $\$75$ *for* 15 *hamburgers, which is a rate of* $\$5$ *per hamburger."*[3]

6.RP.A.3 Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

a. Make tables of equivalent ratios relating quantities with whole–number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.

b. Solve unit rate problems including those involving unit pricing and constant speed. *For example, if it took* 7 *hours to mow* 4 *lawns, then at that rate, how many lawns could be mowed in* 35 *hours? At what rate were lawns being mowed?*

c. Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means $30/100$ times the quantity); solve problems involving finding the whole, given a part and the percent.

d. Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

Solve real-world and mathematical problems involving area, surface area, and volume.

6.G.A.1 Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.

[3]Expectations for unit rates in this grade are limited to non-complex fractions.

© 2015 Great Minds. eureka-math.org
G7-M1-TE-1.3.0-06.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

6.G.A.3 Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.

Focus Standards for Mathematical Practice

MP.1 **Make sense of problems and persevere in solving them.** Students make sense of and solve multi-step ratio problems, including cases with pairs of rational number entries; they use representations, such as ratio tables, the coordinate plane, and equations, and relate these representations to each other and to the context of the problem. Students depict the meaning of constant of proportionality in proportional relationships, the importance of $(0,0)$ and $(1,r)$ on graphs, and the implications of how scale factors magnify or shrink actual lengths of figures on a scale drawing.

MP.2 **Reason abstractly and quantitatively.** Students compute unit rates for paired data given in tables to determine if the data represents a proportional relationship. Use of concrete numbers will be analyzed to create and implement equations, including $y = kx$, where k is the constant of proportionality. Students decontextualize a given constant speed situation, representing symbolically the quantities involved with the formula, $\text{distance} = \text{rate} \times \text{time}$. In scale drawings, scale factors will be changed to create additional scale drawings of a given picture.

Terminology

New or Recently Introduced Terms

- **Constant of Proportionality** (If a proportional relationship is described by the set of ordered pairs that satisfies the equation $y = kx$, where k is a positive constant, then k is called the *constant of proportionality*. For example, if the ratio of y to x is 2 to 3, then the constant of proportionality is $\frac{2}{3}$, and $y = \frac{2}{3}x$.)
- **Miles per Hour** (One *mile per hour* is a proportional relationship between d miles and t hours given by the equation $d = 1 \cdot t$ (both d and t are positive real numbers). Similarly, for any positive real number v, v *miles per hour* is a proportional relationship between d miles and t hours given by $d = v \cdot t$. The unit for the rate, mile per hour (or mile/hour) is often abbreviated as mph.)
- **One-To-One Correspondence Between Two Figures in the Plane (description)** (For two figures in the plane, S and S', a *one-to-one correspondence between the figures* is a pairing between the points in S and the points in S' so that each point P of S is paired with one and only one point P' in S', and likewise, each point Q' in S' is paired with one and only one point Q in S.)

© 2015 Great Minds. eureka-math.org
G7-M1-TE-1.3.0-06.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Looking for Evidence of the Key Shifts

Focus	Evidence
☐ The learning goal(s) of the lesson supports grade-level standard(s).	
Coherence	
☐ The lesson intentionally relates new concepts to students' prior skills and knowledge. ☐ Students set the foundation for future learning. ☐ The teacher reinforces standards that are major work in the grade when teaching supporting or additional content.	
Rigor	
Conceptual Understanding ☐ Students access concepts and ideas from a variety of perspectives. ☐ Students explain mathematical ideas behind a particular concept in a variety of ways. ☐ Students use examples and counterexamples to make and support conjectures that are applied to one problem to multiple situations. ☐ Students create and use a variety of models to analyze relationships. ☐ Students make use of patterns and structure to compose and decompose numbers, shapes, expressions, and equations. **Procedural Skills and Fluency** ☐ Students select tools (e.g., physical objects, manipulatives, drawings, diagrams, algorithms, or strategies) that are relevant and useful for the task or problem. ☐ Students communicate thinking using appropriate vocabulary, symbols, and/or units in precise and accurate ways. ☐ Students look for patterns, generalizations, and shortcuts. ☐ Students are flexible in their use of procedures and skills to solve problems. **Application** ☐ Students decontextualize and contextualize quantities in problem situations. ☐ Students plan and choose a solution pathway when applying their mathematical knowledge to different situations.	

Note: To help educators look for evidence of grade-level-appropriate student engagement in mathematical tasks, these narrative descriptors are adapted from Illustrative Mathematics. (2014, February 12). *Standards for Mathematical Practice: Commentary and elaborations for K–5 and 6-8*. Tucson, AZ: Author. Available at http://commoncoretools.me/2014/02/12/k-5-elaborations-of-the-practice-standards

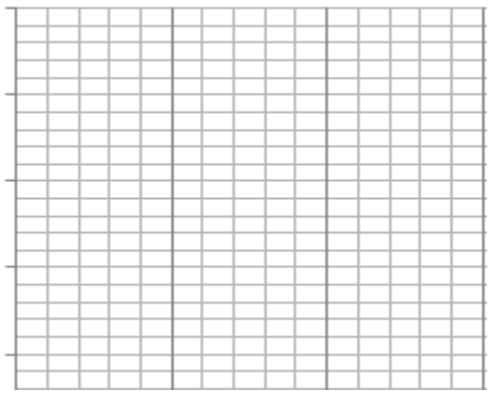
Lesson 6: Identifying Proportional and Non-Proportional Relationships in Graphs

Classwork

Today's Exploratory Challenge is an extension of Lesson 5. You will be working in groups to create a table and graph and to identify whether the two quantities are proportional to each other.

Poster Layout

Use for notes

Problem:	**Table**:
Graph: 	**Proportional or Not? Explanation**:

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from G7-M1-TE-1.3.0-07.2015

This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Gallery Walk

Take notes and answer the following questions:

- Were there any differences found in groups that had the same ratios?
- Did you notice any common mistakes? How might they be fixed?
- Were there any groups that stood out by representing their problem and findings exceptionally clearly?

Poster 1:

Poster 2:

Poster 3:

Poster 4:

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from G7-M1-TE-1.3.0-07.2015

This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Poster 5:

Poster 6:

Poster 7:

Poster 8:

Note about Lesson Summary:

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from G7-M1-TE-1.3.0-07.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Lesson Summary

The plotted points in a *graph of a proportional relationship* lie on a line that passes through the origin.

Problem Set

Sally's aunt put money in a savings account for her on the day Sally was born. The savings account pays interest for keeping her money in the bank. The ratios below represent the number of years to the amount of money in the savings account.

- After one year, the interest accumulated, and the total in Sally's account was $\$312$.
- After three years, the total was $\$340$. After six years, the total was $\$380$.
- After nine years, the total was $\$430$. After 12 years, the total amount in Sally's savings account was $\$480$.

Using the same four-fold method from class, create a table and a graph, and explain whether the amount of money accumulated and the time elapsed are proportional to each other. Use your table and graph to support your reasoning.

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from G7-M1-TE-1.3.0-07.2015

This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Name _____ Date_____

Lesson 6: Identifying Proportional and Non-Proportional Relationships in Graphs

Exit Ticket

1. Which graphs in the gallery walk represented proportional relationships, and which did not? List the group number.

   Proportional Relationship Non-Proportional Relationship

2. What are the characteristics of the graphs that represent proportional relationships?

3. For the graphs representing proportional relationships, what does $(0, 0)$ mean in the context of the given situation?

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from G7-M1-TE-1.3.0-07.2015
16
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Diamond Reflection
Unpacking the EngageNY Lesson

FOCUS: How are the mathematics content standards evident in the lesson?

RIGOR: How did students' engagement in the mathematics reinforce the components of rigor (conceptual understanding, procedural skill and fluency, and application)?

Other thoughts and considerations

COHERENCE: How does this lesson fit into students' larger understanding of proportional relationships?

PLANNING: What instructional strategies and facilitator moves brought out the content and practice standards?

determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4 **Model with mathematics.**
Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5 **Use appropriate tools strategically.**
Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6 **Attend to precision.**
Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7 **Look for and make use of structure.**
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as

Modeling Standards

Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol (★).

What is Modeling?

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

Some examples of such situations might include:

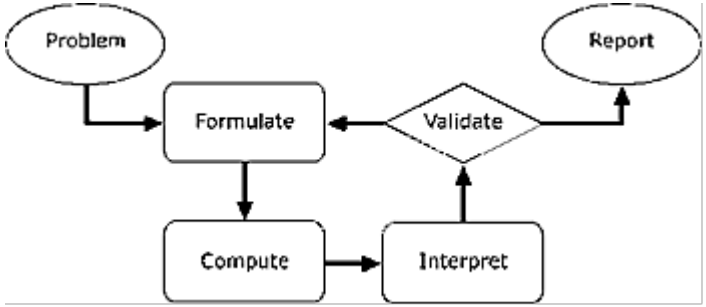
- Estimate how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
- Plan a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Design the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyze the stopping distance for a car.
- Model a savings account balance, bacterial colony growth, or investment growth.
- Engage in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Analyze the risk in situations such as extreme sports, pandemics, and terrorism.
- Relate population statistics to individual predictions.

In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them.

Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.

The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.



In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model—for example, graphs of global temperature and atmospheric CO_2 over time.

Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters which are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems.

Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena (e.g., the behavior of polynomials) as well as physical phenomena).

Standard for Mathematical Practice (SMP) 4: Act One

Exploring Modeling: Notes

Exploring Modeling: Act One

Persistence and effective effort

Use with *Exploring* "Key influences on learning and achievement"

Persistence is continuation of effort and striving in the face of difficulty, opposition, or failure: it is a key characteristic of successful people across professional and academic disciplines. Persistence is evidenced by willingness to continue to try in the face of challenge. For students, this persistence can be a driving force to help them achieve their academic and personal goals.

But where does persistence in the face of adversity come from, and how can educators and parents promote it? Persistence, or **academic resilience**—academic achievement despite a challenging or threatening circumstance in the educational process—is often described as an outcome of high motivation and positive self-beliefs. For example, students with a strong belief in their own capabilities are shown to persist longer through academic difficulty.

Adults play a significant role in helping students develop persistence and apply **effective effort**—proven strategies paired with attitudes and beliefs that enable students to see the greatest benefit. They can reinforce a view of intelligence as malleable, or changeable, provide frequent and specific feedback to students on their academic progress, and encourage students to reflect on their own experiences with overcoming challenges and succeeding.

Educators can foster in students specific skills that are associated with effective effort, such as time management, organizational strategies, and goal setting. When students have a plan for how to deal with challenging academic situations that includes metacognitive strategies, or strategies for examining their own thinking, they are better equipped to continue to try when they struggle.

And classroom practice and routines influence students' effort. Students are more likely to

Copyright 2017 Agile Mind, Inc. ®
Content copyright 2017 Charles A. Dana Center, The University of Texas at Austin

persist in productive ways when teachers:

- Allow time for reflection and discussion.
- Include opportunities for error analysis on assignments and tests.
- Use questioning to uncover confusion as opposed to giving an answer right away.
- Assign tasks that have clear outcomes and rubrics so students can self-assess.
- Praise effort and processes as opposed to outcomes.
- Encourage students to use multiple problem-solving strategies.
- Foster a community of learners in their classrooms.
- Push students to honestly evaluate their struggles and their attributions for success and failure.

Related to this key idea, this course will explore:

- Specific tools and processes for introducing and reinforcing effective effort strategies with your students.
- Research related to self-management, self-regulated learning, grit, and goal-setting.
- How persistence and effective effort are related to students' understanding of the malleability of intelligence, their motivation and self-efficacy beliefs, and the culture of learning in classrooms and schools.

Citation and permission:

Used with permission of Agile Mind, Inc., the article "Persistence and effective effort," which appears in An Educator's Course in Academic Youth Development, by the Charles A. Dana Center and Agile Mind, Inc., available to users of Dana Center/Agile Mind programs in mathematics and science.

To learn more about the Academic Youth Development family of programs, see http://www.agilemind.com/programs/academic-youth-development.

Thinking about Thinking: Self-reflection tool

Use with *Exploring* "Persistence and effective effort"

Use these prompts to help you reflect on the problem–solving experience.

1. When I first read the problem, I thought…
2. I started this problem/task by…
3. My initial strategy got me to this point in the problem…
4. I made a plan when…
5. When I got stuck, I…
6. I knew I was stuck because…
7. The approach I took with this problem (would or would not) work on other problems because…
8. If I were given a similar task, I would do the following differently…
9. What did I learn from this problem that I could use in solving other problems?
10. What have I learned about myself as a problem solver/learner?

Citation and permission:

Used with permission of Agile Mind, Inc., "Thinking About Thinking: Self-Reflection Tool," which appears in the Academic Youth Development programs by the Charles A. Dana Center and Agile Mind, Inc., available to users of Dana Center/Agile Mind programs in mathematics and science.

To learn more about the Academic Youth Development family of programs, see http://www.agilemind.com/programs/academic-youth-development

Copyright 2018 Agile Mind, Inc.™ Content copyright 2018 Charles A. Dana Center, The University of Texas at Austin
Used with permission

Table of Contents[1]

Linear and Exponential Functions

Module Overview 3

Topic A: Linear and Exponential Sequences (**F-IF.A.1**, **F-IF.A.2**, **F-IF.A.3**, **F-IF.B.6**, **F-BF.A.1a**, **F-LE.A.1**, **F-LE.A.2**, **F-LE.A.3**) 12

Lesson 1: Integer Sequences—Should You Believe in Patterns? 14

Lesson 2: Recursive Formulas for Sequences 26

Lesson 3: Arithmetic and Geometric Sequences 35

Lesson 4: Why Do Banks Pay YOU to Provide Their Services? 44

Lesson 5: The Power of Exponential Growth 51

Lesson 6: Exponential Growth—U.S. Population and World Population 61

Lesson 7: Exponential Decay 71

Topic B: Functions and Their Graphs (**F-IF.A.1**, **F-IF.A.2**, **F-IF.B.4**, **F-IF.B.5**, **F-IF.C.7a**) 80

Lesson 8: Why Stay with Whole Numbers? 82

Lessons 9–10: Representing, Naming, and Evaluating Functions 92

Lesson 11: The Graph of a Function 116

Lesson 12: The Graph of the Equation $y = f(x)$ 134

Lesson 13: Interpreting the Graph of a Function 153

Lesson 14: Linear and Exponential Models—Comparing Growth Rates 167

Mid-Module Assessment and Rubric 177
Topics A through B (assessment 2 days, return 1 day, remediation or further applications 2 days)

Topic C: Transformations of Functions (**A-REI.D.11**, **F-IF.C.7a**, **F-BF.B.3**) 198

Lesson 15: Piecewise Functions 199

Lesson 16: Graphs Can Solve Equations Too 213

Lessons 17–20: Four Interesting Transformations of Functions 224

Topic D: Using Functions and Graphs to Solve Problems (**A-CED.A.1**, **A-SSE.B.3c**, **F-IF.B.4**, **F-IF.B.6**, **F-IF.C.9**, **F-BF.A.1a**, **F-LE.A.2**, **F-LE.B.5**) 268

[1]Each lesson is ONE day, and ONE day is considered a 45-minute period.

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from ALG I-M3-TE-1.3.0-08.2015

This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Lesson 21: Comparing Linear and Exponential Models Again 270
Lesson 22: Modeling an Invasive Species Population 280
Lesson 23: Newton's Law of Cooling 287
Lesson 24: Piecewise and Step Functions in Context 295
End-of-Module Assessment and Rubric 306
Topics A through D (assessment 2 days, return 1 day, remediation or further applications 3 days)

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from ALG I-M3-TE-1.3.0-08.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Algebra I • Module 3

Linear and Exponential Functions

OVERVIEW

In earlier grades, students defined, evaluated, and compared functions and used them to model relationships between quantities (**8.F.A.1**, **8.F.A.2**, **8.F.A.3**, **8.F.B.4**, **8.F.B.5**). In this module, students extend their study of functions to include function notation and the concepts of domain and range. They explore many examples of functions and their graphs, focusing on the contrast between linear and exponential functions. They interpret functions given graphically, numerically, symbolically, and verbally; translate between representations; and understand the limitations of various representations.

In Topic A, students explore arithmetic and geometric sequences as an introduction to the formal notation of functions (**F-IF.A.1**, **F-IF.A.2**). They interpret arithmetic sequences as linear functions with integer domains and geometric sequences as exponential functions with integer domains (**F-IF.A.3**, **F-BF.A.1a**). Students compare and contrast the rates of change of linear and exponential functions, looking for structure in each, and distinguishing between additive and multiplicative change (**F-IF.B.6**, **F-LE.A.1**, **F-LE.A.2**, **F-LE.A.3**).

In Topic B, students connect their understanding of functions to their knowledge of graphing from Grade 8. They learn the formal definition of a function and how to recognize, evaluate, and interpret functions in abstract and contextual situations (**F-IF.A.1**, **F-IF.A.2**). Students examine the graphs of a variety of functions and learn to interpret those graphs using precise terminology to describe such key features as domain and range, intercepts, intervals where the function is increasing or decreasing, and intervals where the function is positive or negative (**F-IF.A.1**, **F-IF.B.4**, **F-IF.B.5**, **F-IF.C.7a**).

In Topic C, students extend their understanding of piecewise functions and their graphs including the absolute value and step functions. They learn a graphical approach to circumventing complex algebraic solutions to equations in one variable, seeing them as $f(x) = g(x)$ and recognizing that the intersection of the graphs of $f(x)$ and $g(x)$ are solutions to the original equation (**A-REI.D.11**). Students use the absolute value function and other piecewise functions to investigate transformations of functions and draw formal conclusions about the effects of a transformation on the function's graph (**F-IF.C.7**, **F-BF.B.3**).

Finally, in Topic D, students apply and reinforce the concepts of the module as they examine and compare exponential, piecewise, and step functions in a real-world context (**F-IF.C.9**). They create equations and functions to model situations (**A-CED.A.1**, **F-BF.A.1**, **F-LE.A.2**), rewrite exponential expressions to reveal and relate elements of an expression to the context of the problem (**A-SSE.B.3c**, **F-LE.B.5**), and examine the key features of graphs of functions, relating those features to the context of the problem (**F-IF.B.4**, **F-IF.B.6**).

The Mid-Module Assessment follows Topic B. The End-of-Module Assessment follows Topic D.

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from ALG I-M3-TE-1.3.0-08.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Focus Standards

Write expressions in equivalent forms to solve problems.

A-SSE.B.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★

c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $\left(1.15^{1/12}\right)^{12t} \approx 1.01212t$ *to reveal the approximate equivalent monthly interest rate if the annual rate is* 15%.[2]

Create equations that describe numbers or relationships.

A-CED.A.1[3] Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*★

Represent and solve equations and inequalities graphically.

A-REI.D.11[4] Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.★

Understand the concept of a function and use function notation.

F-IF.A.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

F-IF.A.2 Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

F-IF.A.3[5] Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by* $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ *for* $n \geq 1$.

---

[2]Tasks have a real-world context. As described in the standard, there is an interplay between the mathematical structure of the expression and the structure of the situation such that choosing and producing an equivalent form of the expression reveals something about the situation. In Algebra I, tasks are limited to exponential expressions with integer exponents.
[3]In Algebra I, tasks are limited to linear, quadratic, or exponential equations with integer exponents.
[4]In Algebra I, tasks that assess conceptual understanding of the indicated concept may involve any of the function types mentioned in the standard except exponential and logarithmic functions. Finding the solutions approximately is limited to cases where $f(x)$ and $g(x)$ are polynomial functions.
[5]This standard is part of the Major Content in Algebra I and is assessed accordingly.

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from ALG I-M3-TE-1.3.0-08.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Interpret functions that arise in applications in terms of the context.

F-IF.B.4[6] For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★

F-IF.B.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function $h(n)$ gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★

F-IF.B.6[7] Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★

Analyze functions using different representations.

F-IF.C.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★

a. Graph linear and quadratic functions and show intercepts, maxima, and minima.

b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

F-IF.C.9[8] Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal description). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Build a function that models a relationship between two quantities.

F-BF.A.1[9] Write a function that describes a relationship between two quantities.★

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

---

[6] Tasks have a real-world context. In Algebra I, tasks are limited to linear functions, quadratic functions, square root functions, cube root functions, piecewise functions (including step functions and absolute value functions), and exponential functions with domains in the integers.

[7] Tasks have a real-world context. In Algebra I, tasks are limited to linear functions, quadratic functions, square root functions, cube root functions, piecewise functions (including step functions and absolute value functions), and exponential functions with domains in the integers.

[8] In Algebra I, tasks are limited to linear functions, quadratic functions, square root functions, cube root functions, piecewise functions (including step functions and absolute value functions), and exponential functions with domains in the integers. The focus in this module is on linear and exponential functions.

[9] Tasks have a real-world context. In Algebra I, tasks are limited to linear functions, quadratic functions, and exponential functions with domains in the integers.

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from ALG I-M3-TE-1.3.0-08.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Build new functions from existing functions.

F-BF.B.3[10] Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $k\,f(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Construct and compare linear, quadratic, and exponential models and solve problems.

F-LE.A.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.★

a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals.

b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

F-LE.A.2[11] Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).★

F-LE.A.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★

Interpret expressions for functions in terms of the situation they model.

F-LE.B.5[12] Interpret the parameters in a linear or exponential function in terms of a context.★

Foundational Standards

Work with radicals and integer exponents.

8.EE.A.1 Know and apply the properties of integer exponents to generate equivalent numerical expressions. *For example,* $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$.

[10] In Algebra I, identifying the effect on the graph of replacing $f(x)$ by $f(x)+k$, $k\,f(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative) is limited to linear and quadratic functions. Experimenting with cases and illustrating an explanation of the effects on the graph using technology is limited to linear functions, quadratic functions, square root functions, cube root functions, piecewise functions (including step functions and absolute value functions), and exponential functions with domains in the integers. Tasks do not involve recognizing even and odd functions. The focus in this module is on linear and exponential functions.
[11] In Algebra I, tasks are limited to constructing linear and exponential functions in simple (e.g., not multi-step) context.
[12] Tasks have a real-world context. In Algebra I, exponential functions are limited to those with domains in the integers.

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from ALG I-M3-TE-1.3.0-08.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

8.EE.A.2 Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

Define, evaluate, and compare functions.

8.F.A.1 Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.[13]

8.F.A.2 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.*

8.F.A.3 Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. *For example, the function $A = s^2$ giving the area of a square as a function of its side length is not linear because its graph contains the points $(1,1)$, $(2,4)$, and $(3,9)$, which are not on a straight line.*

Use functions to model relationships between quantities.

8.F.B.4 Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

8.F.B.5 Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

Reason quantitatively and use units to solve problems.

N-Q.A.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

N-Q.A.2[14] Define appropriate quantities for the purpose of descriptive modeling.

N-Q.A.3 Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

[13] Function notation is not required in Grade 8.

[14] This standard is assessed in Algebra I by ensuring that some modeling tasks (involving Algebra I content or securely held content from Grades 6–8) require students to create a quantity of interest in the situation being described.

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from ALG I-M3-TE-1.3.0-08.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Interpret the structure of expressions.

A-SSE.A.1 Interpret expressions that represent a quantity in terms of its context.★

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.*

A-SSE.A.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

Create equations that describe numbers or relationships.

A-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★

A-CED.A.3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*★

A-CED.A.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R.*★

Understand solving equations as a process of reasoning and explain the reasoning.

A-REI.A.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

Solve equations and inequalities in one variable.

A-REI.B.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

Solve systems of equations.

A-REI.C.6[15] Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

Represent and solve equations and inequalities graphically.

A-REI.D.10 Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

[15] Tasks have a real-world context. In Algebra I, tasks have hallmarks of modeling as a mathematical practice (e.g., less-defined tasks, more of the modeling cycle, etc.).

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from ALG I-M3-TE-1.3.0-08.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Focus Standards for Mathematical Practice

MP.1 **Make sense of problems and persevere in solving them.** Students are presented with problems that require them to try special cases and simpler forms of the original problem to gain insight into the problem.

MP.2 **Reason abstractly and quantitatively.** Students analyze graphs of non-constant rate measurements and apply reason (from the shape of the graphs) to infer the quantities being displayed and consider possible units to represent those quantities.

MP.4 **Model with mathematics.** Students have numerous opportunities to solve problems that arise in everyday life, society, and the workplace (e.g., modeling bacteria growth and understanding the federal progressive income tax system).

MP.7 **Look for and make use of structure.** Students reason with and analyze collections of equivalent expressions to see how they are linked through the properties of operations. They discern patterns in sequences of solving equation problems that reveal structures in the equations themselves (e.g., $2x + 4 = 10$, $2(x - 3) + 4 = 10$, $2(3x - 4) + 4 = 10$).

MP.8 **Look for and express regularity in repeated reasoning.** After solving many linear equations in one variable (e.g., $3x + 5 = 8x - 17$), students look for general methods for solving a generic linear equation in one variable by replacing the numbers with letters (e.g., $ax + b = cx + d$). They pay close attention to calculations involving the properties of operations, properties of equality, and properties of inequalities, to find equivalent expressions and solve equations, while recognizing common ways to solve different types of equations.

Terminology

New or Recently Introduced Terms

- **Average Rate of Change** (Given a function f whose domain includes the closed interval of real numbers $[a, b]$ and whose range is a subset of the real numbers, the *average rate of change* on the interval $[a, b]$ is $\frac{f(b)-f(a)}{b-a}$.)
- **Domain** (Refer to the definition of *function*.)
- **Function** (A *function* is a correspondence between two sets, X and Y, in which each element of X is matched[16] to one and only one element of Y. The set X is called the *domain*; the set Y is called the *range*.)
- **Linear Function** (A *linear function* is a polynomial function of degree 1.)
- **Piecewise Linear Function** (Given non-overlapping intervals on the real number line, a (real) *piecewise linear function* is a function from the union of the intervals to the set of real numbers such that the function is defined by (possibly different) linear functions on each interval.)
- **Range** (Refer to the definition of *function*.)

[16] *Matched* can be replaced with *assigned* after students understand that each element of X is matched to exactly one element of Y.

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from ALG I-M3-TE-1.3.0-08.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Looking for Evidence of the Key Shifts

Focus	Evidence
☐ The learning goal(s) of the lesson supports grade level standard(s).	
Coherence	
☐ Students build new understandings onto foundations built in previous years. ☐ Students set the foundation for future learning. ☐ The teacher reinforces standards that are major work in the grade when teaching supporting or additional content.	
Rigor	
Conceptual Understanding ☐ Students access concepts and ideas from a variety of perspectives. ☐ Students explain mathematical ideas behind a particular concept in a variety of ways. ☐ Students use examples and counterexamples to make and support conjectures applied to one problem to multiple situations. ☐ Students create and use a variety of models to analyze relationships in context. ☐ Students make use of patterns and structure to compose and decompose numbers, shapes, expressions, and equations. **Procedural Skills and Fluency** ☐ Students select tools (e.g. physical objects, manipulatives, drawings, diagrams, algorithms, or strategies) that are relevant and useful for the task or problem. ☐ Students communicate thinking using appropriate vocabulary, symbols and/or units in precise and accurate ways. ☐ Students look for patterns, generalizations, and shortcuts. ☐ Students are flexible in their use of procedures and skills to solve problems. **Application** ☐ Students decontextualize and contextualize quantities in problem situations. ☐ Students plan and choose a solution pathway when applying their mathematical knowledge to different situations.	

Note: To help educators look for evidence of grade-level-appropriate student engagement in mathematical tasks, these narrative descriptors are adapted from Illustrative Mathematics. (2014, February 12). *Standards for Mathematical Practice: Commentary and Elaborations for K–5*. Tucson, AZ. Available at http://commoncoretools.me/2014/02/12/k-5-elaborations-of-the-practice-standards

Lesson 22: Modeling an Invasive Species Population

Classwork

Mathematical Modeling Exercise

The lionfish is a fish that is native to the western Pacific Ocean. The lionfish began appearing in the western Atlantic Ocean in 1985. This is probably because people bought them as pets and then dumped them in waterways leading to the ocean. Because it has no natural predators in this area, the number of lionfish grew very quickly and now has large populations throughout the Caribbean as well as along the eastern coastline of the United States and the Gulf of Mexico. Lionfish have recently been spotted as far north as New York and Rhode Island.

The table below shows the number of new sightings by year reported to NAS (Nonindigenous Aquatic Species), which is a branch of the U.S. Geological Survey Department.

Year	Number of New Sightings	Total Number of Sightings
1985	1	
1992	1	
1995	3	
1996	1	
2000	6	
2001	25	
2002	50	
2003	45	
2004	57	
2005	43	
2006	51	
2007	186	
2008	173	
2009	667	
2010	1,342	

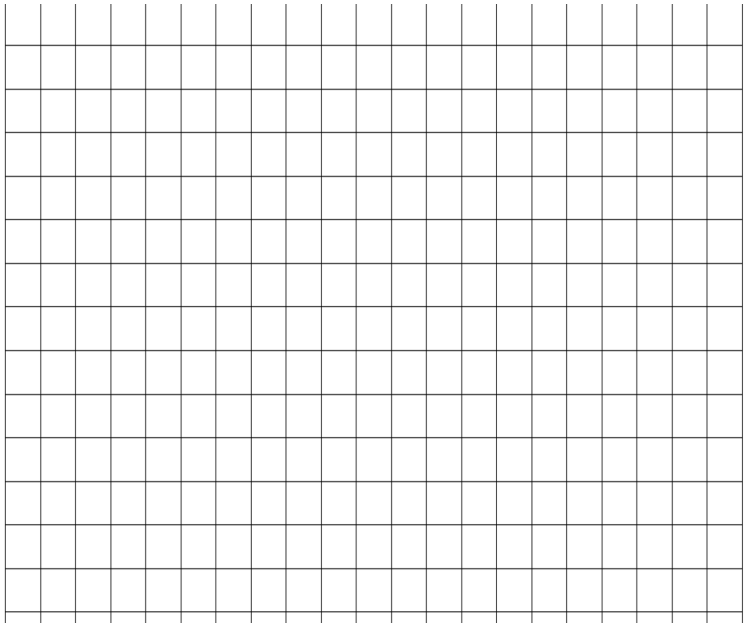
1. Complete the table by recording the total number of sightings for each year.

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from ALG I-M3-TE-1.3.0-08.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

2. Examine the total number of sightings data.
   a) How might you best represent this data? Explain your reasoning.

   b) What pieces of data are important? Why?

3. Make a scatter plot of the year versus the total number of sightings.



4. Explain how the scatter plot supports your answer from Exercise 2a and 2b.

5. On the scatter plot, draw a smooth curve that best fits the data.

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from ALG I-M3-TE-1.3.0-08.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

6. From your table, calculate the average rate of change in the total number of sightings for each of the following time intervals.

   a. 1995–2000 b. 2000–2005 c. 2005–2010

7. How do the average rates of change help to support your argument of whether a linear or an exponential model is better suited for the data?

TURN AND TALK

With a partner, discuss the following:

- What are the critical pieces of data in this problem? Why?

- What does x represent in the context of this situation? What does x = 0 represent?

- What changes can we make to x and to the table to generate a model that we can make more sense of?

8. Use the regression feature of a graphing calculator to find an equation that models the number of lionfish sightings each year.

9. Use your model to predict the total number of lionfish sightings by the end of 2013.

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from ALG I-M3-TE-1.3.0-08.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

10. The actual number of sightings as of July 2013 was 3,776. Does it seem that your model produced an accurate prediction? Explain.

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from ALG I-M3-TE-1.3.0-08.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Problem Set

Another Invasive Species Problem: Kudzu

Kudzu, a perennial vine native to Southeast Asia, now covers a large area of the southern United States. Kudzu was promoted as a forage crop and an ornamental plant when it was introduced to the U.S. at the Philadelphia Centennial Exposition in 1876. Many southern farmers were encouraged to plant kudzu for erosion control from the mid-1930s to the mid-1950s. In 1953, kudzu was removed from the U.S. Department of Agriculture's list of permissible cover plants due to its recognition as an invasive species.

Look up information about kudzu in the U.S. on Wikipedia, and write a short (1- to 2-page) report on the growth of kudzu since its introduction. In your report, choose a function (linear or exponential) to model and graph the growth of kudzu (in hectares) in the U.S. per year over the past half century or so. Remember to cite your sources!

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from ALG I-M3-TE-1.3.0-08.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Handout M3 P-06
Calculator Key Strokes

The following keystrokes apply to the TI-84 family of calculators.

<u>To enter a table of data:</u>

Press **STAT**
Then select **1: Edit** from the list:

```
EDIT CALC TESTS
1:Edit…
2:SortA(
3:SortD(
4:ClrList
5:SetUpEditor
```

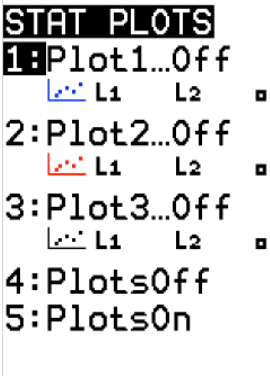
This will bring up lists. Make sure the lists are empty. If the lists are not empty, use the arrows to place your cursor on the column header (L1, L2, etc.) and press **clear** and then **enter**. Now you can enter values into the lists. Use L1 and L2.

L1	L2
0	1
7	2
10	5
11	6
15	12
16	37
17	87
18	132
19	189
20	232
21	283

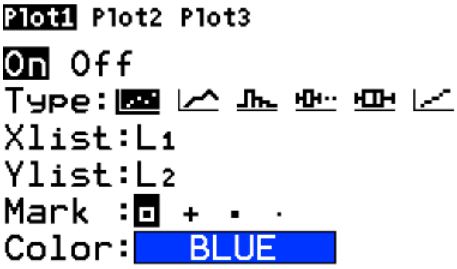
<u>To make a scatterplot:</u>

Press **2nd** then **y=**

Then select **1: Plot 1** from the list:



Press **enter** to turn Plot 1 **On**

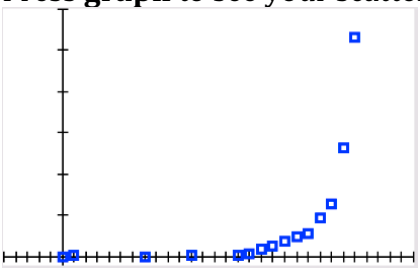


Check the other preferences on the screen above. Make sure the **Xlist** and **Ylist** are the lists where you entered your data.

Press **window** and set an appropriate window to view your scatterplot. Note: to enter a negative value, use the **(–)** symbol next to the **enter** key, not the subtraction symbol.

```
WINDOW
 Xmin=-5
 Xmax=30
 Xscl=1
 Ymin=-100
 Ymax=3000
 Yscl=500
 Xres=1
```

Press **graph** to see your scatterplot.



If other graphs appear, press **y=** and delete any other functions.

To use the regression features:

Press **STAT** and then the right arrow to select **CALC** from the menu

```
EDIT CALC TESTS
1:1-Var Stats
2:2-Var Stats
3:Med-Med
4:LinReg(ax+b)
5:QuadReg
6:CubicReg
7:QuartReg
8:LinReg(a+bx)
9↓LnReg
```

Select the type of regression you want to use to model your scatterplot. (Choose any appropriate model from #3 on. Use the down arrow to see more options.)

For example, to calculate an exponential regression choose **0:ExpReg** from the list above.

```
          ExpReg
 Xlist:L1
 Ylist:L2
 FreqList:
 Store RegEQ:
 Calculate
```

Make sure the correct Lists are given.

You can choose to Store the reqression equation in the **y=** menu by putting your cursor on the Store RegEQ: line then pressing **vars,** then the right arrow to select **Y-VARS**, then **1: Function** and then select which y function you want to use.

Press **enter** twice to calculate the regression equation.

Name _____ Date_____

Lesson 22: Modeling an Invasive Species Population

Exit Ticket

1. For the equation found in Exercise 8, explain the parameters of the equation within the context of the problem.

2. Given each of the following, describe what features in the data or graph make it apparent that an exponential model would be more suitable than a linear model.
   a. The table of data

   b. The scatter plot

   c. The average rates of change found in Exercise 6

3. Use your equation from Exercise 8 to predict the number of lionfish sightings by year 2020. Is this prediction accurate? Explain.

This work is derived from Eureka Math ™ and licensed by Great Minds. ©2015 Great Minds. eureka-math.org
This file derived from ALG I-M3-TE-1.3.0-08.2015
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Diamond Reflection
Unpacking the EngageNY Lesson

Observations: How are the mathematics content standards evident in the lesson?

Reflections: How does student engagement in the mathematics content help develop conceptual understanding, procedural skill and fluency, and application skills?

Decisional: What am I going to do based on my learning?

Interpretations: What instructional strategies and facilitator moves brought out the content and practice standards?

Culture: How does the lesson provide opportunities for students to apply persistence and effective effort strategies?

Assignment:
Teacher Preparation Competencies

In your learning teams,

- Review the math content knowledge and content-pedagogy *Teacher Preparation Competencies*.
- Circle the key content and skills included in each one.
- Explain how the content studied today relates to the competencies and how deep knowledge of the content will assist you in supporting your resident teacher to refine his or her content knowledge and content-specific pedagogy and to meet the competencies.

Mathematics Teacher Competencies

INTRODUCTION

The mathematics teacher preparation competencies identify essential knowledge and skills for teacher candidates seeking certification in birth to kindergarten, pre-kindergarten – third grade, elementary grades 1-5, elementary grades 1-5 integrated to merged, mathematics grades 4-8, mathematics grades 4-8 integrated to merged, mathematics 6-12, and mathematics 6-12 integrated to merged.

CONTENT KNOWLEDGE COMPETENCIES

A. With respect to the academic standards for mathematics for the certification grade band and neighboring grade bands, the teacher candidate, at minimum, demonstrates the following competencies to plan for instruction, teach, and assess student learning:

1. Provides exact, explicit definitions of mathematical ideas and concepts using appropriate mathematical language.
2. Provides precise, accurate, useful descriptions of algorithms and procedures, including descriptions of the accuracy of alternative procedures or algorithms.
3. Exhibits an integrated, functional grasp of mathematical concepts and procedures.
4. Explains concepts and executes procedures flexibly, accurately, efficiently, and appropriately.
5. Models the mathematical dispositions and habits of mind described in the Practice Standards, including precision of language, logical thought, reflection, explanation, and justification.
6. Recognizes and uses the connections between the topics identified in the relevant standards and problems arising in real-world applications.
7. Portrays mathematics as sensible, useful, and worthwhile.

CONTENT PEDAGOGY COMPETENCIES

A. The teacher candidate applies knowledge of mathematical topics and their relationships within and across mathematical domains to identify key mathematical ideas and select or design mathematically sound lesson sequences and units of study that develop students' conceptual understanding, procedural skill and fluency, and ability to solve real-world and mathematical problems.

1. The teacher candidate appropriately sequences content for instruction within a lesson or unit of study and plans appropriate scaffolding to provide opportunities for students to access and master grade-level standards.
2. The teacher candidate anticipates student misconceptions which may arise during a lesson or unit of study, identifies key points in the lesson or unit to check for misconceptions, and identifies appropriate instructional strategies to respond to misconceptions, including but not limited to questioning, whole group discussion, problem sets, instructional tools and representations that make the mathematics of the lesson explicit.
3. The teacher candidate selects or designs standards-based tasks that use varied strategies, including but not limited to real-life applications, manipulatives, models, diagrams/pictures, that present opportunities for instruction and assessment.
4. The teacher candidate selects or designs practice sets that include scaffolding and differentiation of mathematical content to provide opportunities for students to develop and demonstrate mastery.
5. The teacher candidate identifies appropriate student groupings, for example pairs or small groups, to develop students' conceptual understanding, skill, and fluency with mathematical content as well as independent mathematical thinking.

B. The teacher candidate applies understanding of students' mathematical language development to provide regular opportunities during instruction for students to explain their understanding both in writing and orally through classroom conversations.

   1. The teacher candidate explains the connection between students' informal language to precise mathematical language to develop students' ability to use precise mathematical language in their explanations and discussions.

C. The teacher candidate applies understanding of the intersection of mathematical content and mathematical practices to provide regular, repeated opportunities for students to exhibit the math practices while engaging with the mathematical content of the lesson, including but not limited to the following:

   1. Uses appropriate prompting and questioning that allow students to refine their mathematical thinking and build upon one another's understanding of the mathematical content of the lesson.
   2. Poses challenging problems that offer opportunities for productive struggle, encouraging reasoning, problem solving, and perseverance in solving problems in the face of initial difficulty;
   3. Facilitates student conversations in which students are encouraged to discuss each other's thinking in order to clarify or improve their own mathematical understanding;
   4. Provides opportunities for students to choose and use appropriate tools when solving a problem; and
   5. Prompts students to explain and justify work and provides feedback that guides students to produce revised explanations and justifications.

D. The teacher candidate applies knowledge of mathematical topics and their relationships within and across mathematical domains to select or design and use a range of ongoing classroom assessments, including but not limited to diagnostic, formal and informal, formative and summative, oral and written, which determine students' mastery of grade-level standards in order to inform and adjust planning and instruction.

   1. The teacher candidate identifies errors, gaps, and inconsistencies in students' knowledge, skills, and mathematical reasoning to remediate or scaffold students' learning during lesson implementation, using, but not limited to, the following strategies:
      a. oral and written explanations of the elements and structures of mathematics and the meaning of procedures, analogies, and real life experiences;
      b. manipulatives, models, and pictures or diagrams; and
      c. problem sets.
   2. The teacher candidate uses trends in assessment results to plan lesson structure and sequence, instructional strategies, remediation and enrichment opportunities for students.

Module 7 Review

In your team, review the outcomes (p. 3) for Module 5. Share how you anticipate applying what you learned as a mentor.

Exit slip

Use an index card on your table. Write a brief note.

- **3** points that are essential for mentors to remember about the content studied today
- **2** actions you will take as a mentor related to the content studied
- **1** question you have about your responsibilities as a mentor related to the content studied